NON-ABELIAN GAUGE FIELD THEORY
APPLIED TO ELECTRODYNAMICS

ABSTRACT

A theory of electrodynamics is derived from a non-Abelian gauge field theory. In the particular case of O(3) group symmetry, it is shown that the theory gives the $\boldsymbol{B}^{(3)}$ field of radiation as observed in the inverse Faraday effect under all conditions of field matter interaction. It is argued that the conventional U(1) gauge electrodynamics cannot explain the inverse Faraday effect self-consistently because the observable responsible for it, the conjugate product of the radiation field, is not gauge invariant or covariant in U(1). It is however gauge invariant in O(3). Thus $\boldsymbol{B}^{(3)}$ is a fundamental physical observable which is physical, however, only in a gauge that defines $\boldsymbol{B}^{(3)}$.

KEYWORDS: O(3) electrodynamics, $\boldsymbol{B}^{(3)}$ field theory.

1. INTRODUCTION

This paper aims to use the developments in high energy physics over the past thirty five years in an attempt to extend the validity of electrodynamics to non-linear optical effects such as the inverse Faraday effect [1-5], phase-free magnetization by radiation. The overall theoretical framework adopted is gauge theory [6] of the same type as used for the description of quarks, gluons and massive bosons. In Section 2, it is shown that an O(3) gauge theory [7] produces, as a special case, the Gauss law and Faraday law of induction under all conditions. However, the same gauge theory produces a gauge invariant or covariant conjugate product of vector potentials, denoted by $\boldsymbol{A} \times \boldsymbol{A}^*$.

It is argued in Section 3 that this is the radiative observable responsible for the inverse Faraday effect, which occurs in diamagnetic substances such as water as well as in paramagnetic and ferromagnetic substances. In U(1) gauge electrodynamics however, $\boldsymbol{A} \times \boldsymbol{A}^*$ is not gauge invariant or covariant, so the inverse Faraday effect appears to measure the limits of the conventional theory of electrodynamics. The use of gauge theory from high energy physics extends the validity of electrodynamics in such a way as to offer a gauge invariant observable $\boldsymbol{A} \times \boldsymbol{A}^*$ proportional to a magnetic field $\boldsymbol{B}^{(3)}$ that is defined as part of the O(3) symmetry field tensor [8-14].

2. DERIVATION OF THE GAUSS AND FARADAY LAWS FROM GAUGE THEORY

In the gauge theory of high energy physics, there exists a Jacobi identity first derived by Feynman [15, 16],

$$D_\nu \tilde{\boldsymbol{G}}^{\mu\nu} \equiv \boldsymbol{0}, \tag{1}$$

which is valid for all group structures. This theorem states that the covariant derivative, D^μ, of the field tensor $\tilde{\boldsymbol{G}}_{mn}$ is identically zero under all physical conditions. The field tensor is written as a bold capital symbol with subscripts μ**<** in order to demonstrate that there exists an internal gauge space with a well defined group symmetry [16]. When this space has O(3) symmetry, the gauge theory becomes a Yang-Mills theory [17]. In this Section, it is shown that eqn. (1) leads to the Gauss and Faraday laws of electrodynamics, laws which are valid under all conditions. The same procedure produces a gauge invariant $\boldsymbol{A} \times \boldsymbol{A}^*$ [18] as observed in magneto-optics [1-5], a cross-conjugate product of vectors which is proportional to the $\boldsymbol{B}^{(3)}$ field [8-14] of O(3) electrodynamics.

In gauge theory with an O(3) internal symmetry, the field tensor $\boldsymbol{G}_{mn}$, the dual of $\tilde{\boldsymbol{G}}_{mn}$, is defined as [16],

$$\boldsymbol{G}_{mn} \equiv \partial_m \boldsymbol{A}_n - \partial_n \boldsymbol{A}_m + g\boldsymbol{A}_m \times \boldsymbol{A}_n , \tag{2}$$

where the use of bold symbols means that there is an O(3) vector space superimposed on space-time, denoted by the usual μ and **<** [16,17]. The quantity g is a scalar and $\boldsymbol{A}_\mu$ is a vector potential both in the O(3) internal space (a vector space) and in space-time. In this notation, eqn. (1), when written out in full, becomes

$$\partial^n \tilde{\boldsymbol{G}}_{mn} + g\boldsymbol{A}^n \times \tilde{\boldsymbol{G}}_{mn} \equiv \boldsymbol{0}, \tag{3}$$

where $M^<$ is the space-time partial derivative [16]. Equation (3) is a Jacobi identity under all conditions, i.e., in the vacuum as well as in the presence of matter. It is well known that eqns. (1) to (3) are equations of Yang-Mills theory.

In order to derive the Gauss and Faraday laws from eqn. (3), we use the ansatz,

$$\boldsymbol{A}^n \times \tilde{\boldsymbol{G}}_{mn} = \boldsymbol{0}. \tag{4}$$

which implies from eqn. (3) that

$$\partial^n \tilde{\boldsymbol{G}}_{mn} = \boldsymbol{0}. \tag{5}$$

In order to unravel the meaning of eqn. (4), it is helpful to choose an index such as $\mu = 3$ and to write the equation out in full,

$$\boldsymbol{A}^1 \times \tilde{\boldsymbol{G}}_{31} + \boldsymbol{A}^2 \times \tilde{\boldsymbol{G}}_{32} = -\boldsymbol{A}^0 \times \tilde{\boldsymbol{G}}_{30}, \tag{6a}$$

$$\boldsymbol{A}^1 \times \tilde{\boldsymbol{G}}_{21} + \boldsymbol{A}^3 \times \tilde{\boldsymbol{G}}_{23} = -\boldsymbol{A}^0 \times \tilde{\boldsymbol{G}}_{20}, \tag{6b}$$

$$\boldsymbol{A}^2 \times \tilde{\boldsymbol{G}}_{12} + \boldsymbol{A}^3 \times \tilde{\boldsymbol{G}}_{13} = -\boldsymbol{A}^0 \times \tilde{\boldsymbol{G}}_{10}. \tag{6c}$$

It is now assumed that $\tilde{\boldsymbol{G}}_{mn}$ is an electromagnetic field tensor, whose components are Cartesian components of electric and magnetic fields. Thus $\tilde{\boldsymbol{G}}_{31}$ for example, is the ZX component of $\tilde{G}_{mn}$, defined by c / 1),

$$\tilde{G}_{mn} \equiv \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & -E_3 & B_2 \\ B_2 & B_3 & 0 & -E_1 \\ B_3 & -E_2 & B_1 & 0 \end{pmatrix} \tag{7}$$

in contravariant-covariant notation [16] in which $E_1 = -E_X$; $E_2 = -E_Y$; $E_3 = -E_Z$. Similarly $A^1 = A_X$; $A^2 = A_Y$; $A^3 = A_Z$. Therefore the left hand side of eqn. (6a) becomes

$$A_X B_Y - A_Y B_X = -cA^{(0)} B_Z, \tag{8}$$

which can be written as the $\boldsymbol{k}$ component of

$$\boldsymbol{E}^{(1)} \times \boldsymbol{A}^{(2)} = cA^{(0)} \boldsymbol{B}^{(3)}, \tag{9}$$

where

$$\boldsymbol{E}^{(1)} = \boldsymbol{E}^{(2)*} = \frac{E^{(0)}}{\sqrt{2}} (\boldsymbol{i} - i\boldsymbol{j}) e^{if}, \tag{10}$$

$$\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*} = \frac{A^{(0)}}{\sqrt{2}} (i\boldsymbol{i} + \boldsymbol{j}) e^{if}, \tag{11}$$

$$\boldsymbol{B}^{(3)} = B^{(0)} \boldsymbol{k}. \tag{12}$$

Here $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors in the axes X, Y and Z of the Cartesian frame, and N is a dimensionless scalar. The ansatz (4) therefore becomes eqn. (1) in ordinary vector notation.

Similarly, in vector notation, eqn. (5) becomes

$$\nabla \times \boldsymbol{B} = \boldsymbol{0}, \tag{13}$$

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0}. \tag{14}$$

Equation (11) is Gauss's law and eqn. (12) is Faraday's law of induction, both written in S.I. units in the standard vector notation. Of course, there is a wealth of empirical evidence for the equations (11) and (12), which are the fundamental laws of electrodynamics interrelating the fundamental fields $\boldsymbol{E}$ and $\boldsymbol{B}$

under all conditions, in the vacuum as well as for field matter interaction. Clearly, eqns. (10) to (12) must be internally consistent and consistent with eqn. (2) in the O(3) gauge theory.

The self-consistency of eqns. (10) to (12) is shown by the fact that eqn. (10a) is a plane wave solution of eqns. (11) and (12) provided that

$$\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*} = \frac{B^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + j)e^{if}, \tag{13}$$

and provided that

$$f = wt - kZ, \tag{14}$$

where $6 = T/c$ is the wavenumber. Here T is the angular frequency at instant t and position Z. It follows that $\boldsymbol{E}^{(1)} = \boldsymbol{E}^{(2)*}$ and $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ are transverse plane waves. Furthermore, it follows that

$$\boldsymbol{B}^{(3)*} = \boldsymbol{B}^{(3)} = -i\frac{k}{A^{(0)}}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}, \tag{15a}$$

$$\boldsymbol{B}^{(3)*} = \boldsymbol{B}^{(3)} = -i\frac{i}{B^{(0)}}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)}, \tag{15b}$$

where

$$\boldsymbol{B}^{(1)} = \nabla \times \boldsymbol{A}^{(1)}, \qquad \boldsymbol{B}^{(2)} = \nabla \times \boldsymbol{A}^{(2)}. \tag{16}$$

The ansatz (4) of the O(3) gauge theory therefore produces the B Cyclic theorem [8-14],

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = i\boldsymbol{B}^{(0)}\boldsymbol{B}^{(3)*}, \text{ et cyclicum} \tag{17}$$

under all conditions.

This result is internally consistent with eqn. (5) if

$$\partial^n \tilde{\boldsymbol{G}}^{(1)}_{mn} = \boldsymbol{0}, \tag{18a}$$

$$\partial^n \tilde{\boldsymbol{G}}^{(2)}_{mn} = \boldsymbol{0}, \tag{18b}$$

$$\partial^n \tilde{\boldsymbol{G}}^{(3)}_{mn} = \boldsymbol{0}. \tag{18c}$$

Equations (18a) and (18b) are for transverse components and eqn. (18c) for longitudinal components. Equation (15a) is also internally consistent with eqn. (2), which gives

$$\boldsymbol{G}^{(3)*}_{12} = \boldsymbol{B}^{(3)*} = -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}, \tag{19}$$

provided

$$g = \frac{k}{A^{(0)}}. \tag{20}$$

Therefore the electrodynamical equations obtained from O(3) gauge theory are self-consistent, and consistent with the Gauss and Faraday laws. The symmetry of eqns. (2), (15) and (19) shows that there is no $\boldsymbol{E}^{(3)}$ field component. This result is consistent with eqn. (18c), which is

$$\nabla \bullet \boldsymbol{B}^{(3)} = 0, \tag{21}$$

$$\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = \mathbf{0}. \tag{22}$$

3. EMPIRICAL EVIDENCE FOR $\boldsymbol{B}^{(3)} = -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$

Equation (22) is consistent with eqn. (15a) because $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is phaseless. One can say that there is no Faraday induction of an electric field from M$\boldsymbol{B}^{(3)}$/Mt under any condition. This prediction of the O(3) gauge theory has been verified experimentally [19] by pulsing a circularly polarized laser through an evacuated induction coil. No signal was observed. When the experiment was repeated with a pulsed static magnetic field, a signal was observed [19], showing that $\boldsymbol{B}^{(3)}$ is not the curl of a vector potential and is not a magnetic field of U(l) electrodynamics [8-14]. Thus $\boldsymbol{B}^{(3)}$ is not a static U(1) magnetic field, it is a novel radiated field of O(3) electrodynamics and is always defined through the gauge invariant $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ of O(3) electrodynamics,

$$\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} = \frac{c^2}{w^2}\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = \frac{1}{w^2}\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)}. \tag{23}$$

Significantly, $\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} = \boldsymbol{E} \times \boldsymbol{E}^*$ was first introduced phenomenologically by Pershan [20] to predict the inverse Faraday effect, later observed experimentally by van der Ziel *et al*. [21] to have the same Verdet constant as the Faraday effect. Recently, the inverse Faraday effect has been shown to exist [22] in one Dirac electron.

Therefore $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ ($= i\boldsymbol{B}^{(3)}/g$) is a radiative observable at the fundamental level of field to one fermion interaction. It must therefore be physical and gauge invariant or covariant. In classical U(1) electrodynamics however, $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)}$ is neither gauge invariant nor covariant as is well known [22]. It is arbitrary because one can add to $\boldsymbol{A}$ any quantity LM without effecting $\boldsymbol{B}$ / $\mathbf{L} \times \boldsymbol{A}$. Therefore, if we try to apply the rules of classical U(1) gauge transformation to $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, it is inevitably changed in value by LM$^{(1)}$ x LM$^{(2)}$ and becomes arbitrary. This result means that it cannot be a physical quantity of U(1) electrodynamics despite the fact that it is a fundamental radiative observable.

In O(3) electrodynamics, borrowed from well developed gauge theory in high energy physics [16, 17], O(3) gauge transformation leaves $\boldsymbol{G}_{mn}$ invariant or covariant [16], where $\boldsymbol{G}_{mn}$ is defined in eqn. (2).

Therefore $\boldsymbol{B}^{(3)}$ is gauge invariant or covariant in O(3) electrodynamics, but is not so in U(l) electrodynamics. Clearly $\boldsymbol{B}^{(3)}$ is observed empirically whenever $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is so observed. Therefore the inverse Faraday effect is evidence for $\boldsymbol{B}^{(3)}$ and also for O(3) electrodynamics.

DISCUSSION

The above conclusion is quite general because eqn. (1) is a Jacobi identity of gauge field theory for all types of fields under all conditions (field matter interaction and vacuum). This is very useful for clearing the way for unification of all four fundamental fields within non-Abelian gauge field theory. The four fundamental fields are: gravitational, electromagnetic, weak and strong. The introduction of the $\boldsymbol{B}^{(3)}$ field through the observable $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ of the inverse Faraday effect offers a non-Abelian gauge field structure for the electrodynamical sector. The Gauss and Faraday laws of classical electrodynamics are derivable from the more fundamental Jacobi identity (1) with the use of our novel ansatz (4), the B Cyclic theorem [8-14].

The latter therefore plays a crucial role in field unification if this is to proceed on the basis of non-Abelian gauge field theory. One can always attempt to construct a unified four field theory which obeys the Jacobi identity (1) and derive the Gauss and Faraday laws of electrodynamics as special cases. Some efforts in this direction are available in the recent literature [10-14, 23]. In strong field theory for example (quantum chromodynamics), the gauge group is usefully SU(3) [16], which is non-Abelian with structure constants f_{ijk}. The O(3) group used to derive $\boldsymbol{B}^{(3)}$ from the Jacobi identity (1) has structure constants $,_{ijk}$, the Levi Civita symbols. So to unify the strong and electromagnetic sectors, we must relate f_{ijk} to $,_{ijk}$. Loosely speaking, this would show that photons are made up of gluons. It has already been demonstrated [10-14, 23] that the non-Abelian field tensor of O(3) electrodynamics is structured similarly to the Riemann tensor of gravitational theory provided we introduce an anti-symmetric Ricci tensor with anti-symmetric affine connections. This allows $\boldsymbol{G}_{mn}$ of eqn. (2) to become directly proportional [10-14, 23] to an anti-symmetric Ricci tensor, thus expressing O(3) electrodynamics as a gravitational theory. Loosely speaking, this result shows that photons are gravitons. Such a conclusion cannot be reached in U(1) electrodynamics, which is Abelian. Finally, the electromagnetic field can be unified with the weak field by inter-relating the structure constants, as for the strong field. Some suggestions towards this end have been offered already [10-14, 24].

There is a wealth of evidence from high energy physics [25] to suggest that non-Abelian gauge field theory is a good working hypothesis. To date, however, field unification has been hampered by the belief that the electrodynamic sector has U(1) gauge field symmetry, an Abelian symmetry. The gravitational, weak and strong sectors are not Abelian in nature. We have shown that the radiative observable $\boldsymbol{A} \times \boldsymbol{A}^* = \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, responsible for the inverse Faraday effect, indicates that electrodynamics cannot be a U(1) gauge field theory because the radiative observable $\boldsymbol{A} \times \boldsymbol{A}^*$ is neither gauge invariant nor covariant in U(1). It is so, however, in a non-Abelian gauge field theory such as an O(3) symmetry theory that defines the novel and fundamental $\boldsymbol{B}^{(3)}$ field of radiation [8-14] under **all** conditions, including vacuum conditions.

The Abelian gauge field theory that gives rise to U(1) electrodynamics is based on a structure constant of unity [16]. In the U(1) symmetry, the commutator in eqn. (2) is always zero, and the internal gauge space is a scalar space. The experimental limits of U(1) electrodynamics have been reviewed recently by Barrett [17, 26], who has argued that in classical electrodynamics, the potential is observed to have a physical significance and so must be gauge invariant or covariant. In U(1) electrodynamics, it is arbitrary as we have argued. Barrett argues for an SU(2) symmetry theory in electrodynamics, but only under certain conditions. We argue for an O(3) theory under all conditions. The two theories are structurally the same, because SU(2) is the covering group of O(3) [16], but Barrett argues that gauge invariant A_μ potentials are the local manifestations of global constructs. There are many reasons for trying to extend the Maxwell theory in classical electrodynamics [17], and as indicated by non-Abelian gauge theory, the physical nature of the A_μ is the basis for electrodynamics as envisaged in Faraday's electrotonic state [26]. Maxwell built on this concept using quaternions, which transform under SU(2), and not under U(1). The original Maxwellian theory therefore does not have U(1) gauge symmetry; the original equations are based on vector potentials $\boldsymbol{A}$, not on fields such as $\boldsymbol{E}$ and $\boldsymbol{B}$ [26]. The idea of physical A_μ potentials does not occur in Abelian electrodynamics as developed by Heaviside and others in the late nineteenth century. This is what has become known as Maxwell's theory, based on derivative equations for fields, with potentials playing a secondary, non-physical role. In the non-Abelian gauge field theory leading to eqn. (1) however, the potentials are physical and gauge invariant or covariant; and a gauge transformation is a coordinate transformation. Thus, a gauge transformation applied to $\boldsymbol{A} \times \boldsymbol{A}^*$ is a frame rotation that leaves it invariant or covariant [14]. In the U(1) electrodynamics as developed by Heaviside and others [17, 26], however, $\boldsymbol{A} \times \boldsymbol{A}^*$ can never be invariant under a gauge transformation as we have argued already. It becomes arbitrary and unphysical under a U(1) gauge transformation.

Clearly, our use of O(3) in this context is meant to imply a higher order, non-Abelian, symmetry form for the whole of electrodynamics. The Gauss and Faraday laws have an overall O(3) gauge symmetry in our theory, one which allows the $\boldsymbol{B}^{(3)}$ solution in vacuo. In the older U(1) theory, the $\boldsymbol{B}^{(3)}$ field is perpendicular to the plane of definition of U(1) and so is zero by definition. Consequently, the observable $\boldsymbol{A} \times \boldsymbol{A}^*$ is not defined in U(1) theory and its gauge transform rules allow this quantity to be random. Therefore U(1) theory cannot describe the inverse Faraday effect and this was realized by Pershan (20], who constructed the quantity phenomenologically in terms of $\boldsymbol{E} \times \boldsymbol{E}^* = \mathcal{T}^2 \boldsymbol{A} \times \boldsymbol{A}^*$. However, Pershan did not realize that $\boldsymbol{E} \times \boldsymbol{E}^*$ is not gauge invariant, an obvious flaw in his phenomenology. The observable $\boldsymbol{E} \times \boldsymbol{E}^*$ is not gauge invariant in U(1) because U(1) is a linear electrodynamics. Therefore our use of the Jacobi identity, eqn. (1), has the result of giving a self-consistent theory of non-linear and linear electrodynamics. This cannot be done without the $\boldsymbol{B}^{(3)}$ field.

The use of SU(2) by Barrett [17, 26] is similar, but he confines it to certain topological situations in which the A_μ potentials are the local manifestations of global constructs that do not occur in Abelian electrodynamics as developed by Heaviside and his contemporaries. The use of the term **global** by Barrett [17, 26] is meant to convey a non-local property, defined within a volume with boundary conditions. There are no boundary conditions in Heaviside's view of the Maxwell equations, they are four differential equations in fields with no defined boundary conditions in general. The Heaviside electrodynamics can be put in U(1) form because they contain no intrinsic non-Abelian relations such as the proportionality of $\boldsymbol{B}^{(3)}$ to $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$. The U(1) equations are metric independent, i.e., invariant under all diffeomorphism. The topological phase cannot be defined [17, 26], although it is observable. (Similarly, $\boldsymbol{A} \times \boldsymbol{A}^*$ cannot be defined in U(1), although it is observable.) In non-Abelian gauge field theory on the other hand, for example Yang-Mills gauge field theory [16, 17], we are led to inter-related concepts which have no existence in a linear theory such as U(1). For example instantons, solitons, and degenerate vacua with boundary conditions [16]. Barrett argues [17, 26] that in all physically useful situations, the A_μ potentials of electrodynamics have a meaningful physical existence related to the choice of boundary conditions, a choice which determines the transformation group. Similarly, the existence of $\boldsymbol{B}^{(3)}$ in our non-Abelian O(3) gauge theory is linked with boundary conditions in Yang-Mills theory, of which $\boldsymbol{B}^{(3)}$ theory is a special case. For example, $\boldsymbol{B}^{(3)}$ can be thought of as a vortex line [16] supported by an O(3) gauge group topology.

Boundary conditions (and global conditions) enter in to the basic Jacobi identity (1) because it is constructed with covariant derivatives introducing the vector potentials as Feynman's universal influence [16]. The potentials in this view play the same role as the affine connections in general relativity, in which equation (1) is a Bianchi identity. Without the covariant derivatives, the inverse Faraday effect is not described in electrodynamics. Our view is therefore inevitably the same, basically, as that of Barrett [17, 26], because the global or universal influence is introduced as soon as we use covariant derivatives in the Jacobi identity (1). The latter, being an identity, holds in the vacuum as well as in the presence of matter, whereas Barrett [17, 26] confines his consideration to experimental situations where matter is always present. This appears to be a matter of choice of application rather than any difference in theoretical structure. Our use of O(3) is compatible with Barrett's use of SU(2) because the two groups are homomorphic [16], one being the covering group of the other. It is well known that quaternions or spinors have to be used in SU (2) whereas vectors can be used in O(3). There is a mathematical relation between the spinor and vector as is well known. Therefore our choice of O(3) is in line with the original, unmodified, Maxwellian theory, a conclusion which is demonstrated by the fact that $\boldsymbol{B}^{(3)}$ is defined in terms of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$.

Therefore the introduction [8-14] of the $\boldsymbol{B}^{(3)}$ field in vacuo is a direct consequence of the reduction of the Jacobi identity of non-Abelian gauge field theory to the Gauss and Faraday laws using our ansatz (4), the B Cyclic theorem [8-14]. This deduction allows a considerable injection of concepts into electrodynamics borrowed directly from the work in high energy physics of the last four decades, so is a considerable advance in understanding.

By using the ansatz (4), we have effectively confined attention to a classical electrodynamical structure in which the magnetic monopole is zero by construction. More generally, the SU(2) group is central to concepts in high energy physics which can be applied to electrodynamics. Barrett [17, 26] argues that twistors, instantons, magnetic monopole constructs and soliton forms all have pseudo particle SU(2) correspondence, and such concepts can be expected to play a role in classical and quantum electrodynamics provided that the U(1) restriction is lifted. Several results are available already in the literature which indicate that U(1) electrodynamics are incomplete. For example, as clarified by Barrett [26], the Wu-Yang theory attempts to complete the Maxwell theory with a non-integrable, path dependent, phase factor as a physically meaningful quantity demonstrating potential gauge invariance or covariance, and giving an explanation of the Aharonov-Bohm effect with physical potentials. There is a link between the Wu-Yang theory and the observable topological phase [17] on the one hand and with $\boldsymbol{B}^{(3)}$ on the other, both being non-Abelian concepts missing from U(1) electrodynamics of the Heaviside variety. This classical electrodynamics is of course one of the most brilliant developments in natural philosophy, but is now reaching the limits of its applicability, even within a classical context. The $\boldsymbol{B}^{(3)}$ theory is one symptom of this among several others [8-14, 17, 26] and it is simply time to return to Maxwell's original theory, based as it is on quaternions and Faraday's electrotonic state, a prototype physical potential. In view of the fact that Maxwell developed his algebra with Hamilton's quaternions, he developed it in what is now known as an SU(2) form. Heaviside "murdered the potentials" and produced a theory, now known as Maxwell's equations, which expresses classical electrodynamics in terms of four differential equations involving fields: The Gauss and Faraday laws, Coulomb's law, and Ampere's law modified by Maxwell through the latter's introduction of the famous displacement current in vacuo. The latter has been well described by Jackson [27] as a stroke of genius, but is not the only way to introduce a displacement current. Other important suggestions have been made recently by Lehnert and Roy [28] and by Chubykalo and Smirnov-Rueda [29]. It is not always realized, furthermore, that the Maxwellian displacement current does not involve movement of charge, it is a pure vacuum phenomenon, attributing to the vacuum a material-like property. The profound philosophical difference between the Heaviside (U(1)) classical electrodynamics, and those of Maxwell and Faraday, is that the former relegates the potential to a mathematical backwater, where it has remained for over one hundred years despite the various twentieth century attempts by Wheeler, Feynman and many others to give it physical significance again, and bring it into the mainstream of progress in natural philosophy.

To this author, the Heaviside view disintegrates when we are faced with the many phenomena of non-linear optics [8-14], prominent among which is the inverse Faraday effect, a phase-free magnetization produced by circularly polarized laser pulses with the same Verdet constant as the Faraday effect itself. Therefore the inverse effect occurs under the same circumstances and in the same materials as the original, i.e., in diamagnetic substances such as water as well as in paramagnetic substances [21]. Thus $\boldsymbol{A} \times \boldsymbol{A}^*$ is a commonplace radiative observable. It is worth repeating the deduction that the observable $\boldsymbol{A} \times \boldsymbol{A}^*$ becomes **random**, however, when subjected to the usual rules of U(1) gauge transformation, applied individually to A and its complex conjugate $\boldsymbol{A}^*$. The reason is that $\boldsymbol{A}$ transforms to $\boldsymbol{A}$ plus a gradient of a random scalar; and similarly for $\boldsymbol{A}^*$. This is a catastrophe akin to the ultra-violet catastrophe which led to the quantum theory because $\boldsymbol{A} \times \boldsymbol{A}^* = \boldsymbol{E} \times \boldsymbol{E}^*/T^2$ is the radiative **observable** responsible for the inverse Faraday effect, and so must be gauge invariant or, at the least, covariant [16]. In the non-Abelian gauge theory with O(3) symmetry, homomorphic with SU(2) symmetry, the rules for local gauge transformation are fundamentally different, both mathematically and philosophically. The gauge theory of high energy physics borrows concepts from general relativity, so the gauge transform is a frame or coordinate transform. In order for $\boldsymbol{B}^{(3)}$ to be invariant under a gauge transform, we simply rotate the frame around the Z axis, so $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$ are changed but $\boldsymbol{B}^{(3)}$ is not, being in the Z axis itself. Such simple geometrical concepts are not available in Heaviside's point of view, and significantly, Pershan had to propose $\boldsymbol{E} \times \boldsymbol{E}^*$ **phenomenologically** in order to predict the later observed [21] inverse Faraday effect, an effect of non-linear optics to which U(1) electrodynamics, linear electrodynamics, simply do not apply. (If they did, there would have been no need for Pershan to **postulate** $\boldsymbol{E} \times \boldsymbol{E}^*$ as late as 1963 [20], this concept would already have been part of the theory.) It is not clear whether Pershan realized that this concept, a physical radiative observable directly proportional to $\boldsymbol{A} \times \boldsymbol{A}^*$, is not gauge invariant in U(1), and is therefore self-contradictory in U(1). The construct $\boldsymbol{E} \times \boldsymbol{E}^*$ (or $\boldsymbol{B} \times \boldsymbol{B}^*$) is to this day described by the non-linear electrodynamicists as an operator **with no Z component**, although in simple vector algebra, it has a Z component.

Barrett (17] has identified several **classical** phenomena in which it is necessary to regard potentials as physical objects: the Aharonov-Bohm; Altshuler-Aronov-Spivak; topological phase; quantum Hall; de Haas van Alphen and Sagnac effects. In each case, the potentials play a physical role. For example, in the Sagnac effect, gauge invariance of the A_μ is necessary, implying the need for a non-Abelian electrodynamics which in turn implies the existence of the fundamental $\boldsymbol{B}^{(3)}$ field as observed in the inverse Faraday effect. In addition, the Ehrenberg-Siday effect [17], the antecedent of the Aharonov-Bohm effect, requires gauge invariance of A_μ to calculate the refractive index in an electron diffraction experiment. The electron microscope depends on the Ehrenberg-Siday effect, and thus on the physical invariance of A_μ. The inverse Faraday effect and all non-linear magneto optical effects depending on $\boldsymbol{A} \times \boldsymbol{A}^*$ can now be added to the list compiled by Barrett of effects linearly dependent on the potential, and thus not describable by Heaviside's U(1) electrodynamics. The topological phase effects are also non-Abelian in nature and so all these phenomena are inter-related and depend on the $\boldsymbol{B}^{(3)}$ field; inter alia being evidence for the existence of the $\boldsymbol{B}^{(3)}$ field. For example, the topological phase observed [17] by the rotation of the plane of polarization of a probe laser propagating in an optical fibre wound around a cylinder in a helical pattern is directly proportional to the expectation value of the angular momentum of the field, and so it is directly proportional to $\boldsymbol{B}^{(3)}/\boldsymbol{B}^{(0)}$. Therefore the $\boldsymbol{B}^{(3)}$ field senses the topological phase, which is a global concept. Taking all these effects into account, and inter-relating them with non-Abelian gauge field theory, it becomes clearly evident that the $\boldsymbol{B}^{(3)}$ field not only exists theoretically under all conditions, but is a fundamental observable in several phenomena.

Barrett traces the root problem of Heaviside electrodynamics to the fact that every classical, polarized, wave is constituted of two polarized vectorial components and is already a multiply connected field in SU(2) form [17]. He notes that this property (e.g. of a transverse plane wave with $\boldsymbol{i}$ and $\boldsymbol{j}$ components) requires a non-Abelian gauge field theory. This logic leads to the B Cyclic theorem and to the $\boldsymbol{B}^{(3)}$ field, starting from the Jacobi identity (1) of non-Abelian gauge field theory. The latter is in turn consistent with the fact that the product $\boldsymbol{A} \times \boldsymbol{A}^*$ is physical and gauge invariant or covariant only in a non-Abelian gauge field theory. Use of our ansatz (4) reduces the Jacobi identity to the Gauss and Faraday laws, but in an overall non-Abelian structure. The latter is clearly different from the Abelian U(1) structure of Heaviside because the latter does not allow $\boldsymbol{B}^{(3)}$ to be non-zero in the vacuum. This must be carefully bourne in mind because, at first sight, the Gauss and Faraday laws are algebraically identical in the U(1) and O(3) theories. If we do not use ansatz (4), extra terms will appear in general in both laws, as discussed with quaternion and spinor algebra by Barrett [17, 26].

Such terms allow the existence of a non-Abelian magnetic monopole as also discussed by Ryder [16].

It should be noted that the inverse Faraday effect in the presence of matter (for example one electron [22]) is always described in terms of $\boldsymbol{A} \times \boldsymbol{A}^*$, it is a magnetization (or magnetic field strength) proportional to this radiative observable. The coefficient of proportionality in the inverse Faraday effect is in the semi-classical theory of non-linear optics [8-14] the imaginary part of a hyper-susceptibility tensor component which exists in all materials. (Recall that the Faraday effect also exists in all materials and is mediated by the same Verdet constant as the inverse Faraday effect.) Therefore $\boldsymbol{B}^{(3)}$ is observable whenever radiation interacts with matter, and according to the Jacobi identity (1), also exists in the vacuum. Recently, Argyris *et al.* [30] have developed this property into a new metric for the vacuum state, and Ciubotariu [31] has postulated the existence of a longitudinal photo graviton.

Finally, Bearden [32], has noted that the notion of a physically meaningful electromagnetic potential may have widespread consequences in physics, engineering and medicine. He has developed this idea in many directions [32], and in some instances appears qualitatively far in advance of the mathematical notions of non-Abelian gauge theory. If these qualitative but deeply thought out ideas webbed by Bearden [32] are borne out by experiment, then electrodynamics will be changed beyond recognition.

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